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Critical functions for complex analytic maps

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Abstract. Critical functions measure the width of the domain of stability around a given fixed point or an invariant circle for complex analytic and area-preserving maps. We discuss their dependence on the rotation number of the invariant curves and we propose some new methods to determine them based on the existence of critical points and on some properties of quasiconformal maps. By means of the majorant series method some rigorous estimates are given for complex area-preserving maps like the semistandard map and the modulated singular map. In particular we make use of the Brjuno function to interpolate critical maps and we prove that the convergence of the Brjuno function is a necessary and sufficient condition for the existence of an analytic invariant curve of given rotation number. We also discuss the optimality of the rigorous bounds obtained.

1. Introduction

The dynamics of complex analytic mappings (endomorphisms of the Riemann sphere and complex area-preserving maps) has recently been intensively studied both by physicists and mathematicians. These dynamical systems are in fact interesting on their own and display many features of real Hamiltonian systems and area-preserving maps.

Analytic maps exhibit quasiperiodic behaviour with *non-trivial small-scale structure* (Manton and Nauenberg 1983, Widom 1983, Mackay and Percival 1987) at the break-up of the invariant curves which locally foliate the complex plane around the fixed point (Siegel discs) or around an invariant circle (Herman rings). This phenomenon has also been observed in area-preserving (Shenker and Kadanoff 1982, Mackay 1983) and critical circle maps (Feigenbaum *et al* 1982), and in the spectrum of certain almost periodic Schrödinger operators (Ostlund *et al* 1982, Simon 1982).

The question of the existence of an analytic conjugation to the linearised map leads to a *small divisor* problem which was solved by Siegel in 1942 who gave the first proof of convergence of a small divisor series. Siegel made use of the majorant series method and some delicate number-theoretical lemmas, and stressed the importance of Diophantine approximation. His ideas have been further developed by Rüssmann (1967, 1972) and Brjuno (1971, 1972) who weakened the number-theoretical conditions on the rotation number needed in the convergence proof. The ideas of Siegel have recently been generalised and applied to Hamiltonian perturbation theory by Eliasson (1988, 1989) who succeeded in writing an absolutely convergent series expansion for the problem of the existence of quasiperiodic KAM orbits.

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Today it is a well known fact that small divisors are the source of instabilities in the motions of Hamiltonian systems. From Kolmogorov–Arnol'd–Moser (KAM) theory one knows that most invariant circles are preserved under small perturbations of integrable maps. On these invariant circles the dynamics is analytically conjugated to translations with rotation numbers which are strongly irrational, so as to verify, for example, some Diophantine inequality. Numerical investigations (Greene 1979) and some rigorous results (Mather 1984, Mackay and Percival 1985) show that if the strength of the perturbation is large enough KAM circles disappear.

The problem of obtaining accurate estimates of the breakdown threshold for an invariant circle of given rotation number is still basically unsolved, since one lacks a rigorous and computationally effective method.

On the other hand it is generally believed that the value of the parameter at which the perturbation series for a given invariant circle diverges coincides with the breakdown threshold, and at least it certainly gives an extremely good lower bound.

A related problem in the study of complex analytic maps is to give realistic estimates for the radius of convergence (Siegel radius) of the power series which conjugates a given map with a rotation around an irrational indifferent fixed point (Siegel disc) or to estimate the modulus of a Herman ring, i.e. a domain conformally equivalent to an annulus surrounding an invariant circle. Moreover, one would like to study the dependence of these quantities on the arithmetic properties of the rotation number (e.g. on the coefficients of the continued fraction expansion) and on the features of the given map (e.g. the sequence of the Taylor coefficients for a holomorphic map, provided some normalisation condition is fixed). The so-called *critical functions* in fact associate, for a given map, the breakdown threshold of an invariant circle with its rotation number.

In this paper we study this problem for complex maps like polynomial and rational functions, the complex sine-circle map and two complex area-preserving maps introduced by Greene and Percival (1981) and Percival and Vivaldi (1988): the semistandard map (SSM) and the modulated singular map (MSM).

Some recent advances in the study of the Siegel centre problem and Herman rings (Herman 1985, 1987a, Douady 1987, Shishikura 1987, Yoccoz 1988) together with the implementation of Brjuno analysis for the SSM and the MSM allow for a detailed study of the question raised above and for very accurate numerical algorithms for estimating critical functions.

Even if the techniques used are rather specialised and exploit the analytical nature of the problem, some ideas and the spirit of the results should help the understanding of Hamiltonian systems. This is especially true if one believes that the basic nature of the problem is (at least in part) purely number-theoretical. Moreover, the majorant series method is commonly used also in the study of perturbation series in Hamiltonian and symplectic mechanics (Eliasson 1988, 1989, Giorgilli and Galgani 1985, Bazzani *et al* 1989).

We now summarise the paper. In section 2 we give some preliminaries about Siegel discs and the Brjuno condition to be verified by the rotation number for the existence of a Siegel disc. We also briefly describe a remarkable result due to Yoccoz (1988) who proves that the Brjuno condition is a necessary and sufficient condition for the existence of a Siegel disc. In section 3 we describe a new method for obtaining accurate estimates of the Siegel radius, and we prove a rigorous convergence estimate announced in Marmi (1988b, 1989) where the method was first used. The dependence of the Siegel radius on the rotation number and on the degree of the polynomial map considered is numerically and analytically investigated. In section 4 we study Herman rings for a model rational map, the Blaschke fraction, and for the sine-circle map. In section 5 we study two complex area-preserving maps, the SSM and the MSM, and we prove that the Brjuno condition is a sufficient condition for the existence of an invariant circle with a given rotation number. The result concerning the MSM was announced by Malavasi and Marmi (1989). The problem of the optimality of the rigorous estimates proven is also briefly discussed. Finally in the appendix we show that the Brjuno condition is also a necessary condition for the existence of those invariant curves of the SSM which are discussed in section 5.

2. Preliminaries: Siegel discs, Brjuno function and Yoccoz's theorem

Let G denote the group of germs of holomorphic diffeomorphisms of $(\mathbf{C},0)$, and let G_λ denote the set of germs $f \in G$ such that $f'(0) = \lambda \in \mathbf{C}^* \equiv \mathbf{C} \setminus \{0\}$ (we recall that a germ of a function f analytic in some neighbourhood of 0, which we will denote as a germ of $(\mathbf{C},0)$, is the equivalence class $[f]$ under the equivalence relation $f \sim g$ if $f = g$ in some neighbourhood of 0; for more details see Jones and Singerman (1987)). Associated with G and G_λ we have their formal analogues $\hat{G}$ and $\hat{G}_\lambda$ given by the formal power series $\sum_{k=1}^{+\infty} f_k z^k$ respectively with the condition that $f_1 \neq 0$ and $f_1 = \lambda$.

We say that $f \in G_\lambda$ is *linearisable* if and only if f belongs to the conjugacy class of G which contains the rotation $R_\lambda : z \mapsto \lambda z$. Equivalently, f is linearisable if and only if the Schröder functional equation (Schröder 1871)

$$f \circ \Phi = \Phi \circ R_\lambda \tag{2.1}$$

has a unique solution $\Phi \in G_1, \Phi(z) = \sum_{k=1}^{+\infty} \Phi_k z^k$. The problem of the existence of a *formal* linearisation $\Phi \in \hat{G}_1$ is easily solved: a necessary and sufficient condition is $\lambda^m - 1 \neq 0$ for all $m \geq 1$. The coefficients Φ_k are recursively obtained in terms of those of f by matching powers in (2.1): clearly $\Phi_1 = 1$, and for all $n \geq 2$

$$\Phi_n = (\lambda^n - \lambda)^{-1} \sum_{j=2}^{n} f_j \sum_{m_1+\ldots+m_j=n} \Phi_{m_1} \cdots \Phi_{m_j}. \tag{2.2}$$

Whenever $|\lambda| \neq 1$ the formal solution is convergent; thus each germ $f \in G_\lambda$ with $|\lambda| \neq 1$ is linearisable (Poincaré 1879). In this trivial case there exists $c > 0$ such that $|\lambda^n - \lambda|^{-1} \leq c$ for all n, and the convergence of Φ is easily shown by means of the majorant series method.

When $|\lambda| = 1$, i.e. $\lambda = e^{2\pi i\omega}$, $\omega \in \mathbf{R}$, if ω is irrational then the formal solution exists but might well be divergent because of the occurrence of the small divisors $(\lambda^n - \lambda)$ in the recurrence (2.2) for the coefficients of Φ. In fact Cremer (1935) constructed a counterexample exhibiting a germ $f_\lambda \in G_\lambda$ which has a sequence of periodic orbits under the iteration $z_{n+1} = f_\lambda(z_n)$ which accumulates at the fixed point $z = 0$.

If Φ is convergent, f is linearisable and has a *Siegel disc* at $z = 0$ given by the maximal open connected neighbourhood U of $z = 0$ invariant under f and image under Φ of a disc $\boldsymbol{D}_r := \{w \in \mathbf{C} | \; |w| < r\}$. The Siegel disc is foliated into invariant manifolds conformally equivalent to circles with rotation number ω (figure 1).

If r_S denotes the radius of convergence of the power series of Φ, then $U = \Phi(\boldsymbol{D}_{r_S})$ and r_S is called the *Siegel radius* corresponding to f. In fact Siegel (1942) proved that

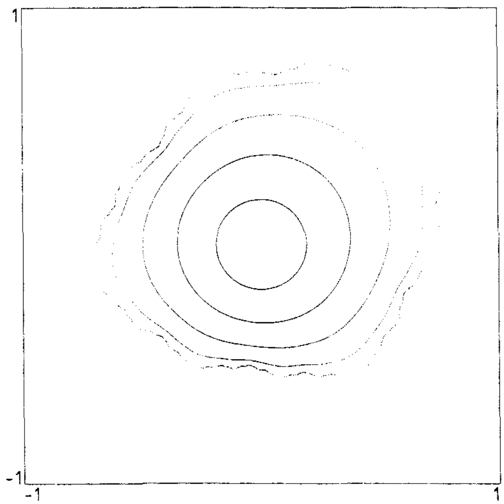

Figure 1. Siegel disc of the quadratic map $z \mapsto e^{2\pi i\omega}z + \frac{1}{2}z^2$ when $\omega = (\sqrt{5}-1)/2$.

if $\omega \in \mathrm{Dioph} := \{x \in \mathbf{R}\setminus\mathbf{Q} | \exists\gamma > 0, \exists\mu \geq 2$ such that $|\omega - p/q| \geq \gamma q^{-\mu}$ for all $p, q \in \mathbf{Z}, q \neq 0\}$, then all the germs $f \in G_\lambda$, where $\lambda = e^{2\pi i\omega}$, are linearisable.

Let $[a_0, a_1, \ldots]$ be the continued fraction expansion of $\omega \in \mathbf{R}\setminus\mathbf{Q}$, recursively determined by $a_j = [1/\omega_j]$ and $\omega_j = \{1/\omega_{j-1}\}$ for all $j \geq 1$, where [ ] and { } denote the integer and fractional parts respectively, $\omega_0 = \omega - [\omega]$ and $a_0 = [\omega]$. The partial fractions

$$\frac{p_k}{q_k} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}}$$

are given by

$$p_k = a_k p_{k-1} + p_{k-2} \qquad q_k = a_k q_{k-1} + q_{k-2}$$

for all $k \geq 0$, with initial data $q_{-2} = p_{-1} = 1$, $q_{-1} = p_{-2} = 0$, and verify the inequality

$$\frac{1}{q_k(q_k + q_{k+1})} < (-1)^k \left(\omega - \frac{p_k}{q_k}\right) < \frac{1}{q_k q_{k+1}}. \tag{2.3}$$

In terms of the growth of the q_k, Cremer's counterexamples were constructed under the assumption that

$$\sup_{k\geq 0}(\log q_{k+1})/q_k = +\infty$$

whilst from Siegel's Diophantine condition it follows that

$$\log q_{k+1} = \mathcal{O}(\log q_k). \tag{2.4}$$

Brjuno (1971, 1972) showed this condition can be weakened; if

$$\sum_{k=0}^{+\infty}(\log q_{k+1})/q_k < +\infty \tag{2.5}$$

then all germs $f \in G_\lambda$ are linearisable.

Very recently Yoccoz has finally proved that the Brjuno condition is in fact also necessary.

Theorem (Yoccoz 1988). Let $\lambda = e^{2\pi i\omega}$, $\omega \in \mathbf{R} \setminus \mathbf{Q}$. All $f \in G_\lambda$ are linearisable if and only if the Brjuno condition (2.5) is verified.

The meaning of Yoccoz's theorem is the following: if ω verifies the Brjuno condition then *all* $f \in G_\lambda$ are linearisable, i.e. *all* germs of holomorphic diffeomorphisms of $(\mathbf{C}, 0)$ such that $f'(0) = \lambda = e^{2\pi i\omega}$. Otherwise one can find at least one element of G_λ which is *not* linearisable. Therefore Yoccoz also proves that the set $S := \{\lambda \in S^1 | G_\lambda$ is a conjugacy class of $G\} = \{\lambda \in S^1 | \Phi$ converges for all $f \in G_\lambda\}$ is invariant under the action of the modular group $\mathrm{PSL}(2, \mathbf{Z})$ (cf Jones and Singerman (1987) for an introduction to the modular group), since if ω is a Brjuno number and $T \in \mathrm{PSL}\ (2, \mathbf{Z})$ then $T\omega$ is also a Brjuno number. We recall that two irrational numbers ω and ω' are *equivalent* if there exists an element $T \in \mathrm{PSL}\ (2, \mathbf{Z})$ such that $\omega = T\omega'$. This definition is equivalent to the condition that the continued fraction expansion of ω and ω' coincide but finitely many terms: $\omega = [a_0, \ldots, a_m, c_0, c_1, \ldots]$, $\omega' = [b_0, \ldots, b_n, c_0, c_1, \ldots]$.

In his proof Yoccoz first introduces a modified continued fraction expansion, then defines a function $B : \mathbf{R} \setminus \mathbf{Q} \to \mathbf{R}^+ \cup \{\infty\}$ which is even, $\mathbf{Z}$-periodic, finite only if it is evaluated at those ω which satisfy the Brjuno condition. Moreover if one normalises $f \in G_\lambda$ imposing the condition that f must be univalent (i.e. holomorphic and injective) in the unit disc D_1, there exists a universal constant C such that if r_S is the Siegel radius corresonding to any such f then

$$|\log r_S^{-1} - B(\omega)| \le C \tag{2.6}$$

where $B(\omega)$ is the Brjuno function.

The function $B(\omega)$ is obtained from the modified continued fraction as follows: let $\langle\ \rangle$ and $\|\ \|$ denote respectively the nearest integer and the distance from the nearest integer of a real number, and let ω be any irrational number. For all $k \ge 1$ we define

$$b_k = \langle \theta_{k-1}^{-1} \rangle \qquad \theta_k = \|\theta_{k-1}^{-1}\| \tag{2.7}$$

with initial data $b_0 = \langle\omega\rangle$, $\theta_0 = \|\omega\|$. Moreover, let $\beta_k = \prod_{i=0}^{k} \theta_k$ if $k \ge 0$, $\beta_{-1} = 1$. The function $B(\omega)$ is defined by

$$B(\omega) := -\sum_{k=0}^{+\infty} \beta_{k-1} \log \theta_k. \tag{2.8}$$

Obviously $B(\omega) = B(\omega + 1) = B(-\omega)$, and for all $\omega \in \mathbf{R} \setminus \mathbf{Q} \cap]0, 1/2]$ it verifies the following functional equation:

$$B(\omega) = -\log \omega + \omega B(\omega^{-1}). \tag{2.9}$$

From this functional equation one can clearly deduce the behaviour of the Brjuno function under the action of the modular group $\mathrm{PSL}(2, \mathbf{Z})$ which is generated by $\omega \mapsto \omega + 1$ and $\omega \mapsto 1/\omega$. Since we will show in section 5 that the Brjuno function can also be used to understand critical functions for area-preserving maps such as the MSM and the SSM, its transformation properties under the action of the modular group should be taken into account if one wishes to understand the scaling properties of critical functions. This problem has been recently investigated by Buric *et al* (1989).

The function $B(\omega)$ converges for all Diophantine numbers and for a class of Liouville numbers. In fact one can show (Yoccoz 1988) that there exists a constant C' independent of ω such that for all $\omega \in \mathbf{R} \setminus \mathbf{Q}$ one has

$$\left| B(\omega) - \sum_{k=0}^{+\infty} \frac{\log q_{k+1}}{q_k} \right| \leq C'. \tag{2.10}$$

We can estimate, for example, that $C' \leq 38$.

Thus, the convergence of $B(\omega)$ is equivalent to condition (2.5).

In figure 2 we have plotted the values of $\exp(-B(\omega))$ at 7000 uniformly distributed random ω in the interval $(0, \frac{1}{2})$. We stress the close similarity to the plot of the Siegel radius as a function of ω reported in section 3.

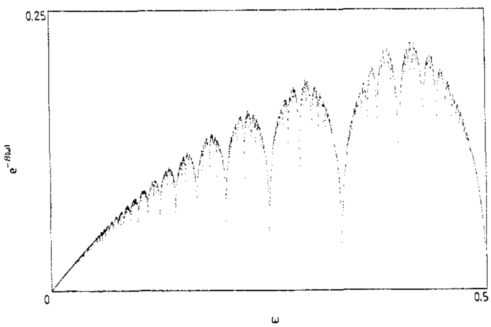

Figure 2. The Brjuno function $\exp(-B(\omega))$ at 7000 uniformly distributed random ω ; $[0, \frac{1}{2}]$.

3. Estimates of the Siegel radius for polynomial maps

From the knowledge of the coefficients Φ_k of the linearisation, applying Hadamard's theorem, one can obtain estimates of the Siegel radius. Upper bounds are also provided by the Bieberbach–De Branges theorem (De Branges 1985) and the area formula for univalent functions (Pommerenke 1975), while lower bounds are usually obtained by means of (computer-assisted) KAM proofs (De La Llave and Rana 1986, Liverani and Turchetti 1986).

All the methods mentioned above have serious shortcomings: upper bounds are rather accurate, but they usually require the computation of several thousand coefficients Φ_k, thus involving a considerable amount of numerical work. Lower bounds obtained analytically are very poor, and can be improved so as to be closer than 10% to the upper bounds only by means of computer-assisted proofs, which again need long computations on the computer.

As we have suggested (Marmi 1988b), much more accurate lower bounds and estimates for the Siegel radius can be obtained by applying the following formula (3.1) (Herman 1987a). Moreover this can be generalised so as to consider Herman rings of rational functions (such as the Blaschke fraction) or entire functions (such as the complex sine-circle map) and compute the modulus of the annulus where the dynamics is conjugated to an irrational rotation (see section 4).

Proposition. Let $f \in G_\lambda$ be linearisable, $\lambda = e^{2\pi i\omega}$. Let U be the Siegel disc of f, $z \in U$, $z = \Phi(w)$, where $w \in \boldsymbol{D}_{r_S}$, $|w| = r < r_S$. Then

$$\lim_{m\to+\infty} \frac{1}{m} \sum_{j=0}^{m-1} \log |f^j(z)| = \log r. \tag{3.1}$$

If p_k/q_k is a partial fraction of ω

$$\left| \frac{1}{q_k} \sum_{j=0}^{q_k-1} \log |f^j(z)| - \log r \right| \leq \frac{1}{q_k} \mathrm{var}(\log |\Phi|) \tag{3.2}$$

where var denotes the variation on the circle $|w| = r$.

Proof. From (2.1) it clearly follows that $f^j(z) = f^j(\Phi(w)) = \Phi(\lambda^j w)$ for all $j \geq 0$ and $w \in \boldsymbol{D}_{r_S}$, thus

$$\frac{1}{m} \sum_{j=0}^{m-1} \log |f^j(z)| = \frac{1}{m} \sum_{j=0}^{m-1} \log |\Phi(\lambda^j w)|.$$

On the other hand, since Φ is a conformal diffeomorphism of $\boldsymbol{D}_{r_S}$ onto U, with $\Phi(0) = 0$, it has neither poles nor zeros but $w = 0$, so that, applying the mean property of harmonic functions to $\log |\Phi(w)/w|$, we find that $\int_0^1 \log |\Phi(re^{2\pi i x})| \, dx = \log r$ for all $r \leq r_S$. Noting that $w \mapsto \lambda w$ is uniquely ergodic on $|w| = r$, by the ergodic theorem we obtain (3.1):

$$\lim_{m\to+\infty} \frac{1}{m} \sum_{j=0}^{m-1} \log |f^j(z)| = \lim_{m\to+\infty} \frac{1}{m} \sum_{j=0}^{m-1} \log |\Phi(\lambda^j w)| = \int_0^1 \log |\Phi(re^{2\pi i x})| \, dx = \log r.$$

For the convergence estimate (3.2) we proceed as follows. Without loss of generality we can suppose that $0 < \omega - p_k/q_k < 1/q_k^2$. Let $\Delta_0 = [0, 1/q_k[$, $\Delta_j =]j/q_k, (j+1)/q_k[$ for $1 \leq j \leq q_k - 1$; thus $\cup_{j=0}^{q_k-1} \Delta_j = [0, 1[$. Consider the action of the translation T_ω by ω on $\boldsymbol{S}^1 = \mathbf{R}/\mathbf{Z}$: $T_\omega^j x_0 = x_0 + j\omega (\mathrm{mod}\, 1) \in \Delta_{jp_k(\mathrm{mod}\, q_k)}$ for all $1 \leq j \leq q_k - 1$. In fact we can assume for simplicity $x_0 = 0$ and we have

$$0 < T_\omega^j x_0 - j\frac{p_k}{q_k} = j\omega - j\frac{p_k}{q_k} \leq \frac{j}{q_k^2} < \frac{1}{q_k}.$$

As p_k and q_k are relatively primes, the sequence $\{\Delta_{jp_k(\mathrm{mod}\, q_k)}\}_{j=0,\dots,q_k-1}$ is the same as $\{\Delta_j\}_{j=0,\dots,q_k-1}$ but with a different ordering and its union gives $\boldsymbol{S}^1$. Finally

$$\begin{aligned}
\frac{1}{q_k} \left| \sum_{j=0}^{q_k-1} \log |f^j(z)| - q_k \log r \right| &= \frac{1}{q_k} \left| \sum_{j=0}^{q_k-1} \log |\Phi(\lambda^j w)| - q_k \int_0^1 \log |\Phi(re^{2\pi i x})| \, dx \right| \\
&= \frac{1}{q_k} \left| \sum_{j=0}^{q_k-1} \left[\log |\Phi(\lambda^j w)| - q_k \int_{\Delta_{jp_k(\mathrm{mod}\, q_k)}} \log |\Phi(re^{2\pi i x})| \, dx \right] \right| \\
&\leq \frac{1}{q_k} \sum_{j=0}^{q_k-1} \sup_{x \in \Delta_{jp_k(\mathrm{mod}\, q_k)}} \Big| \log |\Phi(\lambda^j w)| - \log |\Phi(re^{2\pi i x})| \Big| \\
&\leq \frac{1}{q_k} \mathrm{var}(\log |\Phi|).
\end{aligned}$$

QED

Morever, if f is entire and U is bounded, by taking the radial limit $r \to r_S$, one may show that for a.e. $z \in \partial U$ with respect to the harmonic measure

$$\lim_{m\to+\infty} \frac{1}{m} \sum_{j=0}^{m-1} \log |f^j(z)| = \log r_S. \tag{3.3}$$

We have shown (Marmi 1988b) the accuracy and efficiency of (3.3) to obtain estimates of the Siegel radius when applied to the critical points of f (i.e. to the points z_0 such that $f'(z_0) = 0$).

Numerical experiments show that there is usually a critical point of f on the boundary ∂U of the Siegel disc. Nevertheless Herman has constructed a (pathological) example of a germ f which has a Siegel disc and with the critical point outside its boundary. However no critical point can be contained in U because $f|_U$ is injective, and from the classical theory of Fatou and Julia one knows that ∂U is contained in the closure of the forward orbit $\{f^k(z_0) \mid k \geq 0\}$ of the critical points. Finally Herman himself has proved (Herman 1985) that, for the polynomial maps,

$$f(z) = e^{2\pi i\omega} z + (1/n)z^n \tag{3.4}$$

with $\omega \in$ Dioph and $n \geq 2$, there is a critical point on ∂U.

From now on we will specialise to germs of the form (3.4).

In the special case of $n = 2$ and $\omega \in$Dioph with exponent $\mu = 2$ (irrational numbers with the continued fraction of constant type) Herman has also proved that ∂U is a *quasicircle*, i.e. the image of $\boldsymbol{S}^1$ under a quasiconformal homeomorphism (Herman 1987b). More recently (Shishikura 1989) it has been proven by means of quasiconformal surgery that the same is true for any polynomial map provided that ω is of constant type. In these cases, Φ admits a quasiconformal extension to $|w| = r_S$ and is therefore Hölder continuous (Pommerenke 1975):

$$|\Phi(w_1) - \Phi(w_2)| \leq 4|w_1 - w_2|^{1-\chi} \tag{3.5}$$

for all $w_1, w_2 \in \partial \boldsymbol{D}_{r_S}$, where $\chi \in [0, 1[$ is the Grunsky norm (Pommerenke 1975) associated with the univalent function $g(x) = r_S/\Phi(r_S/x)$ on $|x| > 1$. Therefore mimicking the proof of (3.2) one obtains

$$\left| \frac{1}{q_k} \sum_{j=0}^{q_k-1} \log |f^j(z)| - \log r_S \right| \leq \frac{8}{r_S} \left(\frac{2\pi}{q_k}\right)^{1-\chi} \tag{3.6}$$

and (3.3) now converges to $\log r_S$ *for all* $z \in \partial U$, and in particular when z is the critical point.

In tables 1 and 2 we have reported the estimates of the Siegel radius obtained by applying (3.3) to the orbit $f^{q_k}(z_0)$ of the critical point of the quadratic map when $\omega = (\sqrt{5}-1)/2 = [1, 1, \ldots]$ and $\omega = \sqrt{2}-1 = [2, 2, \ldots]$. The convergence of the formula is rapid, and already after 10^4 iterations the error is about 10^{-5}.

In figures 3 and 4 we have plotted the sums $1/m \sum_{j=0}^{m-1} \log |f^j(z_0)|$ against m for $\omega = (\sqrt{5}-1)/2$ and $\omega = (\sqrt{140^2+4} - 140)/2 = 0.007\,1425\ldots = [140, 140, \ldots]$. Clearly the closer ω is to a rational the longer it takes to the limit (3.3) to reach its asymptotic value. However we again remark that even in this case when $m \geq 10^4$ the relative error

Table 1. Convergence of (3.3) to the Siegel radius: quadratic map $z \mapsto e^{2\pi i\omega}z + \frac{1}{2}z^2$, $\omega = (\sqrt{5}-1)/2$.

$m = q_k$	$\exp\{(1/q_k)\sum_{j=0}^{q_k-1}\log\lvert f^{\prime j}(z)\rvert\}$
1	1.000 0000
2	0.707 1068
3	0.668 7482
5	0.639 2810
8	0.637 6866
13	0.637 6960
21	0.641 1260
34	0.643 6249
55	0.645 9067
89	0.647 4112
144	0.648 4952
233	0.649 1861
377	0.649 6446
610	0.649 9325
987	0.650 1168
1597	0.650 2317
2584	0.650 3041
4181	0.650 3490
6765	0.650 3770
10 946	0.650 3943
17 711	0.650 4051
28 657	0.650 4118
46 368	0.650 4160
75 025	0.650 4185
121 393	0.650 4201
196 418	0.650 4211
317 811	0.650 4217

Table 2. Convergence of (3.3) to the Siegel radius: quadratic map $z \mapsto e^{2\pi i\omega}z + \frac{1}{2}z^2$, $\omega = \sqrt{2}-1$.

$m = q_k$	$\exp\{(1/q_k)\sum_{j=0}^{q_k-1}\log\lvert f^{\prime j}(z)\rvert\}$
2	0.707 1068
5	0.638 5898
12	0.636 9218
29	0.641 4677
70	0.644 9728
169	0.646 7577
408	0.647 5917
985	0.647 9561
2378	0.648 1124
5741	0.648 1783
13 860	0.648 2058
33 461	0.648 2173
80 782	0.648 2221
195 025	0.648 2240
470 832	0.648 2248

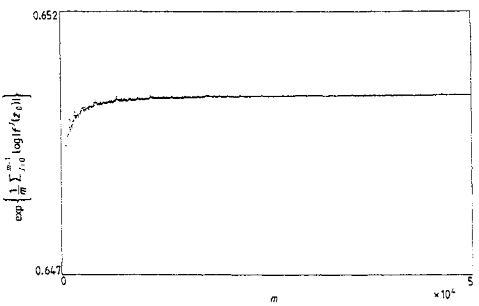

Figure 3. Convergence of (3.3) to the Siegel radius for the quadratic map $z \mapsto e^{2\pi i\omega}z + \frac{1}{2}z^2$ when $\omega = (\sqrt{5}-1)/2 = [1, 1, \ldots]$.

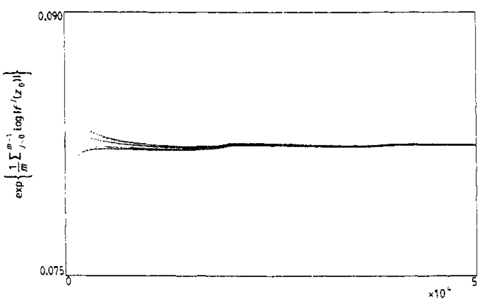

Figure 4. Convergence of (3.3) to the Siegel radius for the quadratic map $z \mapsto e^{2\pi i\omega}z + \frac{1}{2}z^2$ when $\omega = [140, 140, \ldots]$.

is smaller than 10^{-2}. In figure 5 we have plotted the Siegel radius at 7000 uniformly distributed random $\omega \in [0, 1]$ and in figure 6 the ratio $\exp(-B(\omega))/r_S$ is plotted for the same rotation numbers. As (2.10) suggests, this ratio is uniformly bounded away from 0 and $+\infty$, since the Brjuno function has clearly extracted the divergence of $1/r_S$ at rational rotation numbers.

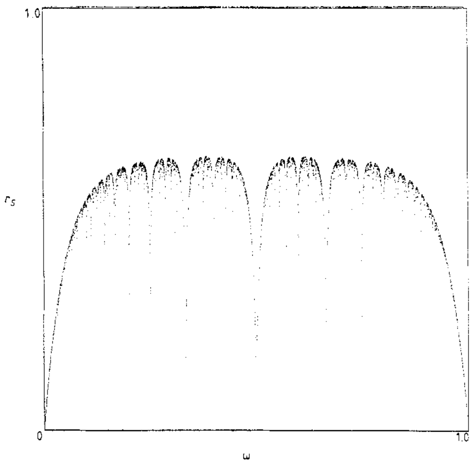

Figure 5. The Siegel radius for the quadratic map $z \mapsto e^{2\pi i\omega}z + \frac{1}{2}z^2$ at the same random ω as figure 2.

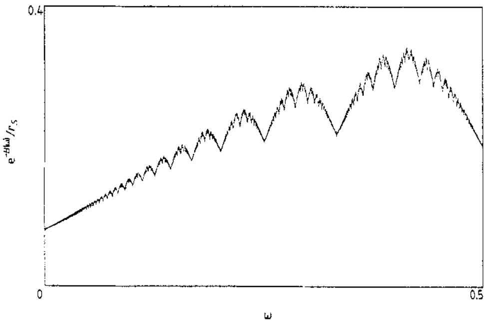

Figure 6. The ratio $\exp(-B(\omega))/r_S$ at 7000 uniformly distributed random ω ; $[0, \frac{1}{2}]$ for the quadratic map $z \mapsto e^{2\pi i\omega}z + \frac{1}{2}z^2$.

All in all this averaging formula gives very good results with a little computer time, and we will use it in the remaining part of this section to study the dependence of the Siegel radius on the arithmetical properties (i.e. the continued fraction) of the rotation number ω for the polynomial maps (3.4). These maps have the critical points which are roots of $z^{n-1} = -e^{2\pi i\omega}$, are univalent on the unit disc $\boldsymbol{D}$ and the linearisation Φ has the following structure:

$$\Phi(w) = \sum_{j=0}^{+\infty} \Phi_{j(n-1)+1} w^{j(n-1)+1} \qquad \Phi_1 = 1 \tag{3.8}$$

i.e. it is a function of w^{n-1}. In fact from the recurrence (2.2) for the coefficients, we have that, for all $k \geq n$,

$$\Phi_k = \frac{1}{n(\lambda^k - \lambda)} \sum_{k_1+\ldots+k_n=k} \Phi_{k_1} \cdots \Phi_{k_n} \tag{3.9}$$

where $\lambda = e^{2\pi i\omega}$, and as $\Phi_l \neq 0$ for all $l < k$ if and only if $l = j(n-1)+1$, we find that $\Phi_k \neq 0$ if and only if $k = k_1 + \ldots + k_n = (j_1 + \ldots + j_n)(n-1) + n = (j_1 + \ldots + j_n + 1)(n-1) + 1$.

This structure of the linearisation clearly explains the periodicity with period $1/(n-1)$ of the Siegel radius as a function of ω. This is clearly illustrated for the cubic map ($n = 3$) in figure 7.

In figures 8, 9 and 10 we have plotted the Siegel radius for the polynomial maps (3.4) of degree $n = 2, 3, \ldots, 10$ by means of 30000 iterates of a critical point when the rotation number $\omega = (\sqrt{p^2+4} - p)/2 = [p, p, \ldots]$ as a function of p. All these irrationals are quadratic, as they are solutions of

$$p + \omega = \frac{1}{\omega}. \tag{3.10}$$

For non-quadratic maps with $n \geq 3$ r_S is not a monotonic function of p, but it has a relative minimum when $p = m(n-1), m \in \mathbb{N}$. This is a consequence of the 'non-genericity' of these maps, and of the structure of the linearisation which is a power

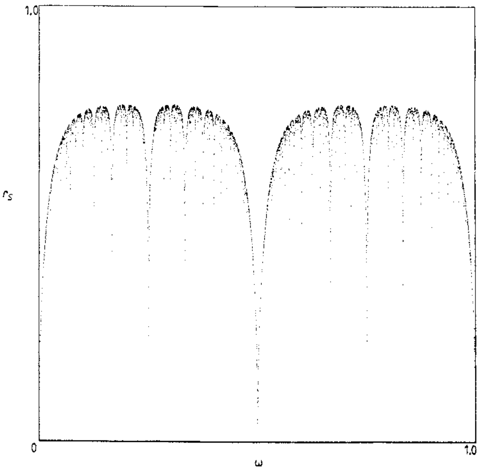

Figure 7. The Siegel radius for the cubic map $z \mapsto e^{2\pi i\omega}z + \frac{1}{3}z^3$.

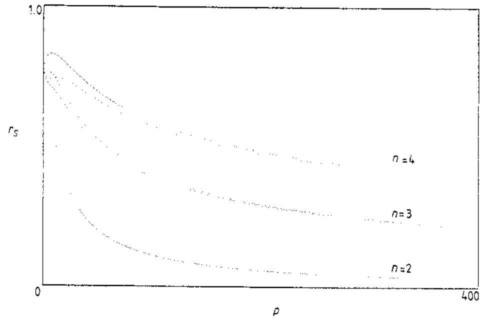

Figure 8. The Siegel radius for polynomial maps $z \mapsto e^{2\pi i\omega}z + \frac{1}{n}z^n$ when $\omega = \sqrt{p^2+4} - p/2 = [p, p, \ldots]$ and $n = 2, 3, 4$.

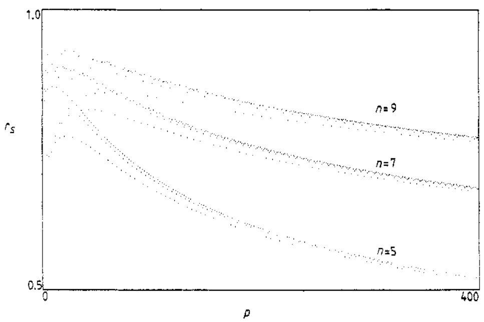

Figure 9. The Siegel radius for polynomial maps $z \mapsto e^{2\pi i\omega}z + n^{-1}z^n$ when $\omega = \sqrt{p^2+4} - p/2 = [p, p, \ldots]$ and $n = 5, 7, 9$.

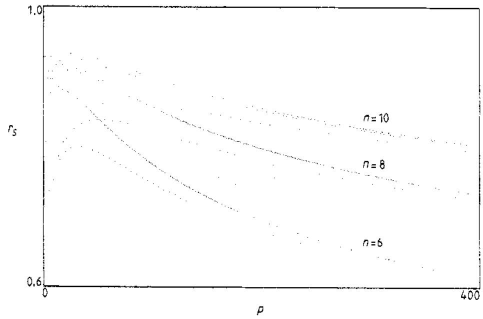

Figure 10. The Siegel radius for polynomial maps $z \mapsto e^{2\pi i\omega}z + n^{-1}z^n$ when $\omega = \sqrt{p^2+4} - p/2 = [p, p, \ldots]$ and $n = 6, 8, 10$.

series in w^{n-1}. Indeed the only small divisors that appear in the recurrence for the coefficients of Φ are

$$|\lambda^{j(n-1)+1} - \lambda| = 2|\sin(j(n-1)\pi\omega) \sim \|j(n-1)\omega\|$$

and amongst these the most important ones occur when $j(n-1) = q_k$, i.e. the denominator of a partial fraction of ω. For general p this is a rare event, but when $p = m(n-1), q_{2k-1} = j_k(n-1)$ for some $j_k \in \mathbf{N}$, i.e. 'half' the small divisors which effectively appear in the recurrence (3.9) coincide with a 'quasi-resonance' due to a partial fraction of ω. In fact $q_1 = p$, $q_3 = p^2 + 1$ and by induction, assuming that $q_{2k-1} = j_k(n-1)$ and $q_{2k-3} = j_{k-1}(n-1)$, one has

$$q_{2k+1} = pq_{2k} + q_{2k-1} = p^2 q_{2k-1} + 2pq_{2k-2} + q_{2k-3} = (n-1)(p^2 j_k + 2q_{2k-2}m + j_{k-1}).$$

When $p \to +\infty$ the Siegel radius converges to zero as expected, approximately according to the power law $r_S \sim (1/p)^{1/(n-1)}$.

If one estimates the Diophantine condition constant γ defined by

$$\gamma := \inf_{l,m \in \mathbf{Z}, m \neq 0} m^2 \left| \omega - \frac{l}{m} \right|$$

by

$$\gamma \sim \frac{1}{p}$$

for the quadratic irrationals solutions of (3.10), one has from KAM proofs (Marmi 1988a) of Siegel's theorem that

$$r_S \geq c\gamma^{1/(n-1)} = c' \left(\frac{1}{p} \right)^{1/(n-1)}$$

where c is some universal constant, which is in agreement with the numerical evidence.

Furthermore, Yoccoz's inequality (2.6) suggests that for generic maps

$$r_S = C(\omega) e^{-B(\omega)}$$

where $C(\omega)$ is uniformly bounded away from 0 and $+\infty$. For the quadratic irrationals solutions of (3.10), by applying (2.9), one immediately has that

$$B(\omega) = -\sum_{k=0}^{+\infty} \omega^k \log \omega = \frac{\log \omega}{\omega - 1}$$

if $p \geq 2$, so that one expects that for the quadratic map, in the limit $p \to +\infty$, i.e. $\omega \to 0$

$$r_S \sim \omega^{1/(1-\omega)} \sim \omega = \frac{1}{2} p \left(\sqrt{1 + \frac{4}{p^2}} - 1 \right) = \frac{1}{2p} + \mathrm{O}\left(\frac{1}{p^2} \right).$$

This estimate is again in agreement with the numerical results.

4. Rational and entire maps of C: Herman rings and estimates of the analyticity strip

If f is a *rational fraction*, $f(z) = P(z)/Q(z)$ where P and Q are polynomials, f defines an endomorphism of the Riemann sphere $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Its degree d is the number of pre-images $f^{-1}(z)$ counting multiplicities, and it is equal to $\max(\deg P, \deg Q)$ if P and Q are relatively primes. From the Riemann–Hurwitz formula f has $2(d-1)$ critical points (counted with multiplicities).

From Sullivan's classification theorem one knows that all the stable regions are eventually cyclic and of one of the following types: attractive and superattractive basins, parabolic basins, *Siegel discs* and *Herman rings* (figure 11). In these last two cases the dynamics is conformally conjugated to an irrational rotation

$$R_\omega : z \mapsto e^{2\pi i \omega} z \tag{4.1}$$

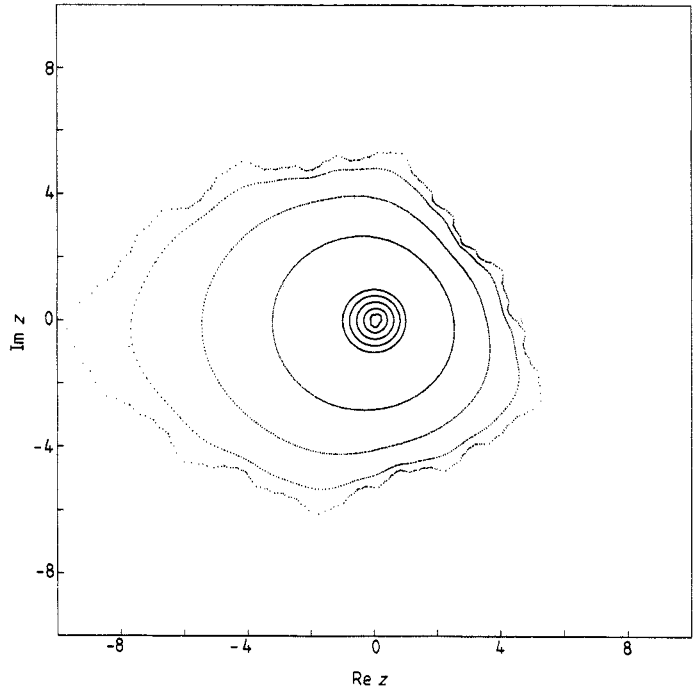

Figure 11. Herman ring associated with the sine-circle map (4.10) for $\varepsilon = 0.2$ and $\omega = (\sqrt{5}-1)/2$.

respectively on the unit disc $\boldsymbol{D}$ and on an annulus $A_r := \{z \in \mathbf{C} | 1/r < |z| < r\}$, $r > 1$. By the maximum principle if f is a polynomial it never has Herman rings, thus we will restrict ourselves from now on to the case $\deg Q \geq 1$.

The boundary ∂V of a Herman ring V, as well as the boundary of a Siegel disc, is contained in the closure of the forward orbit of the critical points. Herman has also proved (1985) that if $\omega \in$ Dioph and $f|_{\overline{V}}$ is injective, then on each of the two boundary components of V there exists at least one critical point of f.

A simple example of a rational map with a Herman ring is provided by the Blaschke fraction

$$f_a : z \mapsto z^2 \frac{z-a}{1-az} \tag{4.2}$$

where $a \in \mathbf{R}$. When $a \in]3, +\infty]$ f_a induces a diffeomorphism of $\boldsymbol{S}^1$, and the critical points different from 0 and ∞ are given by

$$z_\pm = \frac{3 + a^2 \pm \sqrt{a^4 - 10a^2 + 9}}{4a}.$$

Clearly $z_+ z_- = 1$ and $\lim_{a\to 3+} z_+ = \lim_{a \to 3+} z_- = 1$. For each $\omega \in \mathbf{R} \setminus \mathbf{Q}$ one can choose $t \in \mathbf{R}$ such that the rational function

$$f_{a,\omega} : z \mapsto e^{2\pi i t} f_a \tag{4.3}$$

has a rotation number exactly equal to ω.

For each fixed value of ω in the limit $a \to +\infty$ the map (4.3) tends to the rotation R_ω uniformly on every annulus. From Arnol'd's theorem (Arnol'd 1961) on the conjugation of circle diffeomorphisms with rotations and further developments and refinements (Rüssmann 1972, Herman 1979, Yoccoz 1984) one knows that if the denominators q_k of the partial fractions of ω verify the following condition

$$\sum_{k=0}^{+\infty} \frac{\log q_{k+1} \log\log q_{k+1}}{q_k} < +\infty \tag{4.4}$$

then for sufficiently large a the map $f_{a,\omega}$ is analytically conjugated to the rotation R_ω on an annulus $\boldsymbol{A}_r$ containing $\boldsymbol{S}^1$.

Clearly (4.4) is a weaker condition than $\omega \in$ Dioph but stronger than forcing ω to be a Brjuno number (2.5). Until recently it was not known whether the Brjuno condition is also sufficient in this case. By adapting his proof for the case of Siegel discs, in a recent unpublished work Yoccoz has proved that the Brjuno condition is in fact a necessary and sufficient condition for the linearisation of analytic circle diffeomorphisms close to an irrational rotation, whilst for the general case the rotation number must fulfill an additional set of rather technical conditions needed for implementing a reduction procedure to the previous case.

For each fixed value of a one has a critical function $K(\omega, a)$ which is just the radius $\boldsymbol{A}_r$ of the maximal annulus on which $f_{a,\omega}$ is conjugated to the rotation (i.e. the conformal image of the Herman ring V through the conjugacy). By adapting the argument given in the last section one has again an efficient tool for computing this critical function.

If we map the annulus $\boldsymbol{A}_r$ on the strip $\boldsymbol{I}_\eta := \{\theta \in \mathbf{C} \| \operatorname{Im}\theta| \le \eta\}$, with $\eta = \log r$, we have that the map $f_{a,\omega}$ is conjugated to the irrational translation

$$T_\omega : \theta \mapsto \theta + \omega \tag{4.5}$$

through an analytic map $\mathrm{e}^{\mathrm{i}\Phi} : \boldsymbol{I}_\eta \to V$:

$$f_{a,\omega} \circ \mathrm{e}^{\mathrm{i}\Phi} = \mathrm{e}^{\mathrm{i}\Phi} \circ T_\omega. \tag{4.6}$$

For all $z \in V$ one has, therefore,

$$z = \mathrm{e}^{\mathrm{i}\Phi(\theta)} = \mathrm{e}^{\mathrm{i}(\theta + \varphi(\theta))} \tag{4.7}$$

where $\theta \in \boldsymbol{I}_\eta$ and $\varphi := \Phi - \textit{identity}$ is a periodic function with zero average.

From (4.6) it clearly follows that, for all $j \ge 1$,

$$\begin{aligned} \frac{1}{m}\sum_{j=0}^{m-1} \log\left|f^j_{a,\omega}(z)\right| &= \frac{1}{m}\sum_{j=0}^{m-1} \log\left|\mathrm{e}^{\mathrm{i}\Phi(\theta+j\omega)}\right| \\ &= \mathrm{Re}\frac{1}{m}\sum_{j=0}^{m-1}\left[\log\left(\frac{\mathrm{e}^{\mathrm{i}\Phi(\theta+j\omega)}}{\mathrm{e}^{\mathrm{i}(\theta+j\omega)}}\right) + \mathrm{i}(\theta + j\omega)\right]. \end{aligned}$$

Replacing $\theta = \xi + \mathrm{i}\delta$ with $\xi, \delta \in \mathbf{R}$, $|\delta| < \eta$, and using the fact that the irrational translation T_ω is uniquely ergodic, since φ has zero mean we obtain

$$\lim_{m\to+\infty} \frac{1}{m}\sum_{j=0}^{m-1} \log\left|f^j_{a,\omega}(z)\right| = -\delta. \tag{4.8}$$

If one chooses the initial point $z = z_-$, i.e. a critical point of the map, and makes use of the fact that when ω is Diophantine it is conjectured (Shishikura 1987) that the critical points belong to ∂V and that ∂V is a quasicircle, by the same argument given in the previous section one can extend the validity of (4.8) to all the points z on the boundary, including the critical point, so that

$$\lim_{m\to+\infty} \frac{1}{m} \sum_{j=0}^{m-1} \log \left| f^{j}_{a,\omega}(z_-) \right| = \eta. \tag{4.9}$$

In figure 12 we have plotted η at 2000 uniformly distributed random ω when $a = 20$, computed by means of (4.9) applied to the first 50 000 iterations of the critical point. We remark the close similarity with the other critical function plots in this article. In figure 13 the ratio $\exp(-B(\omega))/\eta$ is plotted at the same value of a and for the same ω. The picture shows clearly that the Brjuno condition is a sufficient condition for the existence problem of Herman rings for the Blaschke fraction (4.2) in agreement with Yoccoz's work.

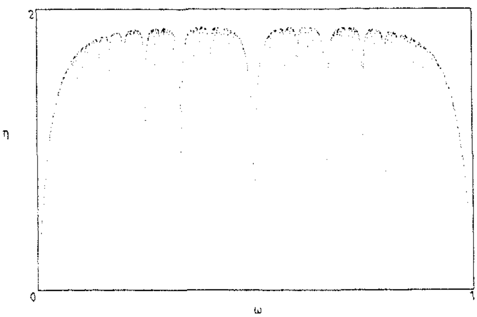

Figure 12. The width of the Herman ring of the Blaschke fraction (4.2) when $a = 20$ at 2000 values of ω ; $[0, 1]$.

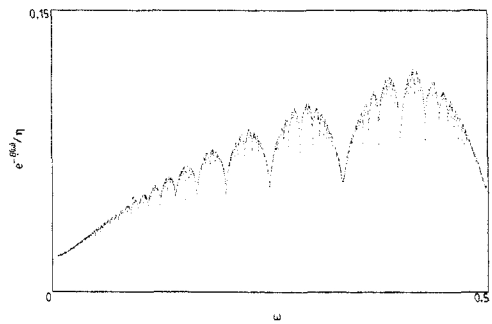

Figure 13. The ratio $\exp(-B(\omega))/\eta$ at 2000 uniformly distributed random ω ; $[0, \frac{1}{2}]$ for the Blaschke fraction (4.2) when $a = 20$.

The same analysis can be carried for the sine-circle map

$$g_{\varepsilon,\omega} : z \mapsto z + t + \varepsilon \sin(z) \tag{4.10}$$

where $z \in \mathbf{C}$ and t is chosen in such a way that ω is the rotation number of $g_{\varepsilon,\omega}$. Again, from Arnol'd's and Yoccoz's theorems one knows that for ε small enough $g_{\varepsilon,\omega}$ is analytically conjugated to T_ω on a strip I_η around the real axis, provided that ω verifies the Brjuno condition (2.5).

For $0 \leq \varepsilon < 1$, if one maps the real line onto S^1 by means of the exponential mapping $w = e^{iz}$, (4.10) induces an analytic diffeomorphism of S^1 and has a Herman ring V surrounding it. The critical points are solutions of the complex equation

$$1 + \varepsilon \cos z = 0$$

from which it readily follows that $z_\pm = u \pm iv$, where

$$u = (2k+1)\pi \quad k \in \mathbf{Z} \qquad \text{and} \qquad v = \log \frac{1 + \sqrt{1-\varepsilon^2}}{\varepsilon}. \tag{4.11}$$

Clearly $\lim_{\varepsilon\to 0+} \operatorname{Im} z_\pm = 0$ and one can numerically verify that the two images $w_\pm = e^{iz_\pm}$ of the critical points indeed each belong to one of the two connected components of the boundary ∂V of the Herman ring associated with (4.10). The argument which leads to (4.8) also establishes that for all $w = e^{iz} \in V$, if z is the image of $\theta = \xi + i\delta \in \boldsymbol{I}_\eta$ under the conjugacy

$$\lim_{m\to+\infty} \frac{1}{m} \sum_{j=0}^{m-1} \log \left| e^{ig^j_{\varepsilon,\omega}(z)} \right| = \lim_{m\to+\infty} \frac{1}{m} \sum_{j=0}^{m-1} \operatorname{Im} g^j_{\varepsilon,\omega}(z) = \delta. \tag{4.12}$$

Moreover if one assumes that the critical points belong to the boundary of the Herman ring, and that it is a quasicircle, by taking radial limits one finds

$$\lim_{m\to+\infty} \frac{1}{m} \sum_{j=0}^{m-1} \operatorname{Im} g^j_{\varepsilon,\omega}(z_-) = \eta \tag{4.13}$$

where η is again the logarithm of the radius r of the annulus to which the Herman ring is conformally equivalent.

In figure 14 we have plotted η as a function of ε when the rotation number is kept fixed with $\omega = (\sqrt{5}-1)/2$ computed by means of (4.13) through the sum of the first 46 368 iterates of the critical point z_-. As expected η is monotonously decreasing with ε and vanishes when $\varepsilon = 1$, i.e. when the critical points $z_\pm$ reach the real axis: this corresponds in fact to the situation when the Herman ring V is reduced to $\boldsymbol{S}^1$ only, since the two components of ∂V existing for $\varepsilon < 1$ now have critical points (and therefore all their images) in common with $\boldsymbol{S}^1$. We also remark that for the same reason when $\varepsilon = 1$ the map (4.10) acting on $\boldsymbol{S}^1$ is not a diffeomorphism any more (critical circle map). For more information about the critical sine-circle map we refer the reader to Shenker (1982) and Feigenbaum *et al* (1982).

5. Complex area-preserving maps

Consider an area-preserving map and denote by K the perturbation parameter, so that when $K = 0$ the phase space is completely foliated into invariant circles. Each invariant circle is uniquely determined by its rotation number ω. From converse KAM theory (Mather 1984, Mackay and Percival 1985) one knows that there exists a critical value $K = K(\omega)$ at which the invariant curve with rotation number ω is destroyed.

The critical function $K = K(\omega)$ has been studied for some complex area-preserving maps by various authors (Greene and Percival 1981, Percival 1982, McGarr and Percival 1984, Percival and Vivaldi 1988, Malavasi and Marmi 1989). These maps show most of the relevant features of real maps, but the study of perturbation series which parametrise the invariant circles is considerably simplified both algorithmically (recursive relations for the coefficients are very simple) and numerically (the series are absolutely convergent, so that, quite differently from *real* area-preserving maps, no subtle cancellations are responsible for their convergence).

We will show in this section using two models, the *modulated singular map* (MSM)

$$y_{n+1} = y_n + \frac{K e^{2\pi i n\omega}}{x_n - 1} \qquad x_{n+1} = x_n + y_{n+1} \tag{5.1}$$

amd the *semistandard map* (SSM)

$$y_{n+1} = y_n + \mathrm{i}K\mathrm{e}^{\mathrm{i}x_n} \qquad x_{n+1} = x_n + y_{n+1} \tag{5.2}$$

that for complex area-preserving maps the Brjuno function also gives a good approximation for the critical function. More precisely we will first prove that convergence of $B(\omega)$ is a sufficient condition for the existence of an invariant circle in the MSM and the SSM. Then we will make use of Yoccoz's theorem to conjecture that for both these maps $K(\omega) \approx \exp[-2B(\omega)]$, and we will verify this result on the SSM (as for the MSM we refer to Malavasi and Marmi 1989).

Assume that $\omega \in \mathbf{R} \setminus \mathbf{Q}$ is a 'strongly irrational' rotation number, so that a corresponding invariant circle exists for $0 \le K \le K(\omega)$ (for the MSM we consider invariant circles with a rotation number equal to the frequency of the modulation). Then the map (MSM or SSM) is analytically conjugated to the rotation

$$z_{n+1} = \mathrm{e}^{2\pi\mathrm{i}\omega} z_n \qquad z_n = K\mathrm{e}^{\mathrm{i}\theta_n} \tag{5.3}$$

where $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ is an angle variable parametrising the invariant circle. This means that there exists a function $\Phi(z)$, holomorphic in the disc $\boldsymbol{D}_{K(\omega)}$, such that for the MSM

$$x_n = \Phi(z_n)$$

$$y_n = x_n - x_{n-1} = \Phi(z_n) - \Phi(z_{n-1}) \tag{5.4}$$

and for the SSM

$$\Phi(z_n) = \mathrm{i}(x_n - \theta_n)$$

$$y_n = x_n - x_{n-1} = -\mathrm{i}[\Phi(z_n) - \Phi(z_{n-1})] + 2\pi\omega. \tag{5.5}$$

The conjugation function $\Phi(z)$ can therefore be expanded into a convergent power series $\Phi(z) = \sum_{n=1}^{+\infty} \Phi_n z^n$, and must verify the functional equation

$$(\Phi(z) - 1)(\delta^2\Phi)(z) = z \qquad \text{MSM} \tag{5.6}$$

and

$$(\delta^2\Phi)(z) = -z\exp[\Phi(z)] \qquad \text{SSM} \tag{5.7}$$

where

$$(\delta^2\Phi)(z) = \Phi(\mathrm{e}^{2\pi\mathrm{i}\omega}z) - 2\Phi(z) + \Phi(\mathrm{e}^{-2\pi\mathrm{i}\omega}z). \tag{5.8}$$

In order to find the recurrence which defines the coefficients of $\Phi(z)$ we define

$$D_n := |2\sin(\pi n\omega)|^2 \qquad n \ge 1 \tag{5.9}$$

which gives the (small) divisors sequence for both MSM and SSM.

By matching powers in (5.6) and (5.7), one easily finds that for the MSM

$$\Phi_1 = \frac{1}{D_1} \qquad \Phi_n = \frac{1}{D_n}\sum_{l=1}^{n-1} \Phi_l \Phi_{n-l} D_{n-l} \qquad n \ge 2 \tag{5.10}$$

and for the SSM

$$\Phi_1 = \frac{1}{D_1} \qquad \Phi_n = \frac{1}{D_n}\sum_{j=1}^{n-1} \frac{1}{j!} \sum_{k_1+\ldots+k_j=n-1} \Phi_{k_1}\ldots\Phi_{k_j} \qquad n \ge 2. \tag{5.11}$$

In both cases the critical function $K(\omega)$ is obtained by means of Hadamard's formula: since all the Φ_n are positive real numbers we can drop the absolute value and

$$K(\omega) = \left(\limsup_{n\to\infty} \Phi_n^{1/n}\right)^{-1}. \tag{5.12}$$

The main analytical result of this section is summarised by the following.

Theorem. A sufficient condition for the existence of an invariant circle in both the MSM and the SSM is Brjuno's condition (2.5)

$$\sum_{k=0}^{+\infty} \frac{\log q_{k+1}}{q_k} < +\infty.$$

Moreover one has the following inequalities:

$$\log K(\omega) \geq -\log 2(1+\sqrt{2}) - 4\sum_{k=0}^{+\infty} \frac{\log q_{k+1}}{q_k} \qquad \text{MSM} \tag{5.13}$$

$$\log K(\omega) \geq -\log \frac{21}{4} - 4\sum_{k=0}^{+\infty} \frac{\log q_{k+1}}{q_k} \qquad \text{SSM.} \tag{5.14}$$

By adapting an argument given by Yoccoz one can also show that if the Brjuno condition is violated, i.e. $B(\omega) = +\infty$, then the series (5.11) diverges, thus proving that the Brjuno condition is *necessary and sufficient* for the existence of the analytic invariant curves of the SSM which admit the parametrisation given by (5.3), (5.5) and (5.11). This is done in the appendix.

Proof. Clearly (5.13) and (5.14) imply the first statement of the theorem. From (5.12)

$$-\log K(\omega) \leq \sup_{n\geq 1} \frac{1}{n} \log \Phi_n$$

so that in order to show (5.13) and (5.14) it suffices to prove that

$$\frac{1}{n} \log \Phi_n \leq C + 4\sum_{k=0}^{+\infty} \frac{\log q_{k+1}}{q_k}$$

where $C = \log 2(1+\sqrt{2})$ for the MSM and $C = \log \frac{21}{4}$ for the SSM.

The proof is based on the majorant series method of Cauchy. For all $n \geq 1$ let

$$\varepsilon_n := |\sin(\pi n\omega)|^2. \tag{5.15}$$

If $\|.\|$ denotes the distance from the nearest integer, $\|y\| := \min_{p\in\mathbb{Z}} |y+p|$, since $2x \leq \sin(\pi x) \leq \pi x$ for all $x \in [0, \frac{1}{2}]$, one immediately has that

$$1 \geq \varepsilon_n \geq 4\|n\omega\|^2. \tag{5.16}$$

According to Siegel's ideas (1942) we introduce the following sequences:

$$\sigma_1 = 1 \qquad \sigma_n = \sum_{l=1}^{n-1} \sigma_l \sigma_{n-l} \qquad n \geq 2 \tag{5.17}$$

and

$$\delta_1 = \frac{1}{\varepsilon_1} \qquad \delta_n = \frac{1}{\varepsilon_n} \max_{1\leq j\leq n-1} \delta_j \delta_{n-j} \qquad n \geq 2 \tag{5.18}$$

which will be used to obtain a majorant series of the conjugation function of the MSM, and for the same purpose, with respect to the SSM conjugation function, we define

$$\sigma'_1 = 1 \qquad \sigma'_n = \sum_{j=2}^{n} \sum_{k_1+\ldots+k_j=n} \sigma'_{k_1} \cdots \sigma'_{k_j} \qquad n \geq 2 \tag{5.19}$$

and

$$\delta'_1 = \frac{1}{\varepsilon_1} \qquad \delta'_n = \frac{1}{\varepsilon_n} \max_{2\leq j\leq n} \max_{k_1+\ldots+k_j=n} \delta'_{k_1} \cdots \delta'_{k_j} \qquad n \geq 2. \tag{5.20}$$

With these definitions we have the following lemmas.

Lemma 1. The coefficients Φ_n of the power series expansion of the MSM conjugation function verify the inequality

$$\Phi_n \leq \sigma_n \delta_n \qquad \text{for all } n \geq 1 \tag{5.21}$$

and for the SSM

$$\Phi_n \leq \sigma'_n \delta'_n \qquad \text{for all } n \geq 1. \tag{5.22}$$

Proof of lemma 1. First we prove (5.21). From the definitions (5.10), (5.17) and (5.18), since $D_n = 4\varepsilon_n$, $\Phi_1 \leq 1/\varepsilon_1 = \delta_1\sigma_1$; by induction

$$\Phi_n = \frac{1}{\varepsilon_n}\sum_{l=1}^{n-1} \Phi_l \Phi_{n-l}\varepsilon_{n-l} \leq \frac{1}{\varepsilon_n}\sum_{l=1}^{n-1} \Phi_l \Phi_{n-l} \leq \frac{1}{\varepsilon_n}\sum_{l=1}^{n-1} \sigma_l \sigma_{n-l}\delta_l\delta_{n-l}$$

$$\leq \left(\frac{1}{\varepsilon_n}\max_{1\leq l\leq n-1} \delta_l\delta_{n-l}\right)\left(\sum_{l=1}^{n-1}\sigma_l\sigma_{n-l}\right) = \sigma_n\delta_n.$$

To prove (5.22) we first define the sequence $b_1 = 1/\varepsilon_1$,

$$b_n = \frac{1}{\varepsilon_n}\sum_{j=2}^{n} \sum_{k_1+\ldots+k_j=n} b_{k_1}\ldots b_{k_j} \qquad n \geq 2 \tag{5.23}$$

and we show that from (5.11) one has

$$\Phi_n \leq b_n \qquad \text{for all } n \geq 1.$$

In fact $\Phi_1 \leq 1/\varepsilon_1 = b_1$ and, since $\Phi_1 = 1/D_1 \geq \frac{1}{4}$, by induction we have

$$\Phi_n \leq \frac{1}{4\varepsilon_n}\sum_{j=1}^{n-1} \sum_{k_1+\ldots+k_j=n-1} \Phi_{k_1}\ldots\Phi_{k_j} \leq \frac{\Phi_1}{\varepsilon_n}\sum_{j=1}^{n-1} \sum_{k_1+\ldots+k_j=n-1} \Phi_{k_1}\ldots\Phi_{k_j}$$

$$= \frac{1}{\varepsilon_n}\sum_{j=2}^{n} \sum_{k_1+\ldots+k_j=n, k_j=1} \Phi_{k_1}\ldots\Phi_{k_j} \leq b_n.$$

From the definitions (5.19) and (5.20) of σ'_n and δ'_n one has $b_1 = \sigma'_1\delta'_1$ and one can immediately check by induction that $b_n \leq \sigma'_n\delta'_n$. This completes the proof of lemma 1.

QED

By this lemma, to prove our theorem it suffices to show that

$$\tfrac{1}{n}\log\sigma_n + \tfrac{1}{n}\log\delta_n \leq \log 2(1+\sqrt{2}) + 4\textstyle\sum_{k=0}^{+\infty} \frac{\log q_{k+1}}{q_k} \qquad \text{MSM}$$
$$\tfrac{1}{n}\log\sigma'_n + \tfrac{1}{n}\log\delta'_n \leq \log \tfrac{21}{4} + 4\textstyle\sum_{k=0}^{+\infty} \frac{\log q_{k+1}}{q_k} \qquad \text{SSM.}$$

The contribution from σ_n and σ'_n is the trivial constant term. In fact the key idea of Siegel's method is that these sequences keep track of the contribution to the growth rate of Φ_n coming from the structure of the recurrence, i.e. the algorithm, disregarding the small divisor problem, i.e. setting $D_n = 1$ in (5.10) and (5.11). Thus one finds that the origin of the divergence of the series is in fact purely number-theoretical (see also the appendix where this is proved for the SSM), in contrast to what happens, for example, for Birkhoff series of symplectic maps (Bazzani *et al* 1989). The sequences δ_n and δ'_n extract the small divisors contribution from the series.

By the very definition of σ_n and σ'_n, if we let $f(z) = \sum_{n=1}^{+\infty} \sigma_n z^n$ and $g(z) = \sum_{n=1}^{+\infty} \sigma'_n z^n$ we find that f and g verify the functional equations

$$f(z) = z + (f(z))^2$$

and

$$g(z) = z + \frac{(g(z))^2}{1 - g(z)}$$

which can be readily solved

$$f(z) = \frac{1 - \sqrt{1 - 4z}}{2}$$

and

$$g(z) = \frac{1 + z - \sqrt{1 - 6z + z^2}}{4}.$$

Therefore f converges for $|z| < 1/4$ and g converges for $|z| < 1/7$, and by Cauchy's estimate

$$\sigma_n \leq 4^n \max_{|z| \leq 1/4} |f(z)| \leq \frac{1 + \sqrt{2}}{2} 4^n \tag{5.24}$$

and

$$\sigma'_n \leq 7^n \max_{|z| \leq 1/7} |g(z)| \leq \tfrac{3}{4} 7^n. \tag{5.25}$$

We now consider δ_n and δ'_n. Here we essentially repeat Brjuno's arguments (Brjuno 1971 and 1972): we also refer to Herman (1987a) and Pöschel (1986) for some very readable expositions and for applications of Brjuno's method to the Siegel theorem.

In (5.18) and (5.20) the maximum is attained for some decomposition

$$\delta_n = \frac{1}{\varepsilon_n} \delta_{j_n} \delta_{n-j_n} \qquad \text{where } 1 \leq j_n \leq n-1 \tag{5.26}$$

and

$$\delta'_n = \frac{1}{\varepsilon_n} \delta'_{k_1^*} \ldots \delta'_{k_{j^*}^*} \qquad \text{where } 2 \leq j^* \leq n, k_1^* + \ldots k_{j^*}^* = n. \tag{5.27}$$

Decomposing $\delta_{j_n}, \delta_{n-j_n}$ and $\delta'_{k_1^*}, \ldots, \delta'_{k_{j^*}^*}$ in the same manner, and proceeding like this we will finally obtain some well defined decomposition

$$\delta_n = \prod_{k=1}^{l(n)} \varepsilon_{i_k}^{-1} \qquad \text{where } \varepsilon_{i_1} = \varepsilon_n, 1 \leq i_2 \leq \ldots \leq i_{l(n)} \leq n-1 \tag{5.28}$$

and

$$\delta'_n = \prod_{k=1}^{l'(n)} \varepsilon_{i'_k}^{-1} \qquad \text{where } \varepsilon_{i'_1} = \varepsilon_n, 1 \leq i'_2 \leq \ldots \leq i'_{l'(n)} \leq n-1. \tag{5.29}$$

Lemma 2.

$$l'(n) \leq l(n) = 2n - 1. \tag{5.30}$$

Moreover $l(1,n) := \operatorname{card}\{k = 1, \ldots, l(n) \mid i_k = 1\} = l'(1,n) := \operatorname{card}\{k = 1, \ldots, l'(n) \mid i'_k = 1\} = n$.

Proof of lemma 2. As $\delta_1 = \delta'_1 = 1/\varepsilon_1$ one has $l(1,n) = l'(1,n) = l(1) = l'(1) = 1$. By induction, for $n \geq 2$ from (5.26) one has

$$l(1,n) = l(1,j_n) + l(1,n-j_n) = n$$

$$l(n) = 1 + l(j_n) + l(n-j_n) = 1 + 2j_n - 1 + 2n - 2j_n - 1 = 2n - 1$$

and from (5.27)

$$l'(1,n) = l'(1,k_1^*) + \ldots + l'(1,k_{j^*}^*) = k_1^* + \ldots + k_{j^*}^* = n$$

$$l'(n) = 1 + l'(k_1^*) + \ldots + l'(k_{j^*}^*) \leq 2(k_1^* + \ldots + k_{j^*}^*) - j^* + 1 \leq 2n - 1$$

since $j^* \geq 2$. This completes the proof of lemma 2. QED

We now consider the function $\Omega : \mathbf{N} \setminus \{1\} \to \mathbf{R}^+$ defined as follows: for all $m \geq 2$

$$\Omega(m) := \min_{1 \leq j \leq m-1} \|j\omega\|^2. \tag{5.31}$$

Clearly Ω is non-increasing and, since $\omega \in \mathbf{R} \setminus \mathbf{Q}$, $\lim_{m \to +\infty} \Omega(m) = 0$.

By the 'law of best approximation' of irrational numbers from their partial fractions (we refer, for instance, to Schmidt 1980, p 21) if $1 \leq q \leq q_k$, $(p,q) \neq (p_k, q_k)$ and $k \geq 1$, where p_k/q_k is a partial fraction (2.3) of ω, $|q\omega - p| > |q_k\omega - p_k|$. Moreover if $(p,q) \neq (p_{k-1}, q_{k-1})$, $|q\omega - p| > |q_{k-1}\omega - p_{k-1}|$. Therefore Ω is piecewise constant, so that $\Omega(q_{k-1}+1) = \Omega(q) = \Omega(q_k) > \Omega(q_k+1)$ for all $q_{k-1}+1 \leq q \leq q_k$, and by (2.3) one has

$$\Omega(q_k) = \min_{1 \leq j \leq q_k - 1} \left(\min_{p \in \mathbf{Z}} |j\omega + p| \right)^2 \geq |q_{k-1}\omega - p_{k-1}|^2 \geq \frac{1}{4q_k^2}. \tag{5.32}$$

The main idea in Brjuno's method is to count the number of terms in the decompositions (5.28) and (5.29) of δ_n and δ'_n which are smaller than $4\Omega(m)$ for a given $m \geq 2$ (the factor 4 is due to (5.16)).

Lemma 3. Let $N_m(n) := \mathrm{card}\{k = 1, \ldots, l(n) \text{ in } (5.28) \mid \varepsilon_{i_k} < 4\Omega(m)\}$ and $N'_m(n) := \mathrm{card}\{k = 1, \ldots, l'(n) \text{ in } (5.29) \mid \varepsilon_{i'_k} < 4\Omega(m)\}$. Then

$$N_m(n) = N'_m(n) = 0 \qquad \text{if } n < m$$

$$N_m(n) \leq 2\left[\frac{n}{m}\right] - 1 \qquad N'_m(n) \leq 2\left[\frac{n}{m}\right] - 1 \qquad \text{if } n \geq m \tag{5.33}$$

where [ ] denotes the integer part.

Proof of lemma 3. We prove the statement for $N'_m(n)$, since the proof for $N_m(n)$ follows the same scheme.

By (5.29), if $n < m$, $m \geq i'_k + 1$ for all $k = 1, \ldots, l'(n)$. Thus from (5.16) we have

$$\varepsilon_{i'_k} = |\sin(\pi i'_k \omega)|^2 \geq 4\|i'_k\omega\|^2 \geq 4\Omega(i'_k + 1) \geq 4\Omega(m)$$

because Ω is not increasing. Therefore $N'_m(n) = 0$.

If $n = m$, for all $k = 2, \ldots, l'(n)$, $i'_k \le n-1 = m-1$, so that by the same argument used above

$$\varepsilon_{i'_k} \ge 4\Omega(i'_k + 1) \ge 4\Omega(m).$$

One is only left with the possibility that $\varepsilon_{i'_1} = \varepsilon_n < 4\Omega(m)$, and $N'_m(m) \le 1$.

We now proceed by induction on n. Let $n > m$: two cases are possible.

Case 1 : $\varepsilon_n \ge 4\Omega(m)$. Then by (5.27) and the induction hypothesis

$$N'_m(n) = N'_m(k_1^*) + \ldots + N'_m(k_{j^*}^*) \le 2\left[\frac{k_1^*}{m}\right] + \ldots + 2\left[\frac{k_{j^*}^*}{m}\right] - j^* \le 2\left[\frac{n}{m}\right] - 2$$

as $k_1^* + \ldots + k_{j^*}^* = n$, $[x] + [y] \le [x+y]$ and $j^* \ge 2$.

Case 2 : $\varepsilon_n < 4\Omega(m)$. Then

$$N'_m(n) = 1 + N'_m(k_1^*) + \ldots + N'_m(k_{j^*}^*) \le 1 + 2\left[\frac{n}{m}\right] - 2 = 2\left[\frac{n}{m}\right] - 1.$$

This completes the proof of lemma 3. QED

We can now complete the proof of the theorem. We define the sequences of index sets $I(0) := \{k = 1, \ldots, l(n) \text{ in (5.28) } | 4\Omega(q_1) \le \varepsilon_{i_k} < 1\}$, $I(m) := \{k = 1, \ldots, l(n) \text{ in (5.28) } | 4\Omega(q_{m+1}) \le \varepsilon_{i_k} < 4\Omega(q_m)\}$, and similarly $I'(0)$ and $I'(m)$ by replacing (5.28) with (5.29) in the definition. The sequence $(q_k)_{k=1}^{+\infty}$ is the sequence of the denominators of the partial fractions of ω. Clearly $\cup_{m=0}^{+\infty} I(m) = \cup_{m=0}^{+\infty} I'(m) =]0,1]$; by lemma 3 if $m \ge 1$

$$\operatorname{card} I(m) \le N_{q_m}(n) \le 2\left[\frac{n}{q_m}\right] - 1 \qquad \operatorname{card} I'(m) \le N_{q_m}(n) \le 2\left[\frac{n}{q_m}\right] - 1 \tag{5.34}$$

and

$$\operatorname{card} I(0) \le 2n - 1 \qquad \operatorname{card} I'(0) \le 2n - 1.$$

Thus, by (5.28) and (5.32)

$$\frac{1}{n}\log\delta_n = \sum_{k=1}^{l(n)} \frac{1}{n}\log\varepsilon_{i_k}^{-1} \le \sum_{m=0}^{+\infty} \frac{1}{n}\operatorname{card} I(m)\log\frac{1}{4\Omega(q_{m+1})} \le 4\sum_{m=0}^{+\infty}\frac{1}{q_m}\log q_{m+1} \tag{5.35}$$

and analogously from (5.29) and (5.32) one has

$$\frac{1}{n}\log\delta'_n = \sum_{k=1}^{l'(n)} \frac{1}{n}\log\varepsilon_{i'_k}^{-1} \le 4\sum_{m=0}^{+\infty}\frac{1}{q_m}\log q_{m+1}. \tag{5.36}$$

Combining (5.35) together with (5.24) one immediately gets for the MSM

$$\sup_{n\ge1}\frac{1}{n}\log\Phi_n \le \sup_{n\ge1}\left(\frac{1}{n}\log\sigma_n + \frac{1}{n}\log\delta_n\right) \le \log\frac{1+\sqrt{2}}{2} + \log 4 + 4\sum_{k=0}^{+\infty}\frac{\log q_{k+1}}{q_k}$$

which completes the proof of (5.13); from (5.36) and (5.25) we find for the SSM that

$$\sup_{n\ge1}\frac{1}{n}\log\Phi_n \le \sup_{n\ge1}\left(\frac{1}{n}\log\sigma'_n + \frac{1}{n}\log\delta'_n\right) \le \log\frac{3}{4} + \log 7 + 4\sum_{k=0}^{+\infty}\log\frac{q_{k+1}}{q_k}$$

which establishes (5.14). The theorem is proved. QED

As it is rather evident, the reader can adapt the proof given above of estimates (5.13) and (5.14) to the recurrence (2.2) for the conjugacy of the Siegel problem, thus obtaining that

$$\log r_S \geq \hat{C} - 2\sum_{k=0}^{+\infty} \frac{\log q_{k+1}}{q_k} \tag{5.37}$$

where $\hat{C}$ is a constant independent of ω.

In the Siegel problem the small divisors $|\lambda^n - \lambda|$ have the form

$$\hat{\varepsilon}_n = \sqrt{\varepsilon_{n-1}} = |\sin(\pi(n-1)\omega)|. \tag{5.38}$$

due to the fact that the functional equation (2.1) is a first-difference equation whereas (5.7) and (5.8) are second-difference equations. Their contribution to the growth rate of the majorant series of the linearisation, which comes through a sequence $\hat{\delta}_n$ which is defined in (5.20) replacing ε_n with $\hat{\varepsilon}_n$, is clearly half of that coming from the divisors ε_n for the MSM and the SSM.

Because of this close analogy, the result (2.8) of Yoccoz (1988) for the Siegel problem, together with (2.10), suggests that the factor 4 on the RHS of (5.13) and (5.14) can be replaced by 2, so that one is naturally led to conjecture that

$$K(\omega) = \frac{e^{-2B(\omega)}}{C_M(\omega)} \tag{5.39}$$

for the MSM, and

$$K(\omega) = \frac{e^{-2B(\omega)}}{C_S(\omega)} \tag{5.40}$$

for the SSM, where $C_M(\omega)$ and $C_S(\omega)$ are positive continuous functions, bounded away from 0 and $+\infty$ uniformly in ω.

It has been shown (Malavasi and Marmi 1989) that the conjecture is numerically well verified for the MSM. For what attains the SSM in figure 15 we have plotted the critical function $K(\omega)$ at 5000 uniformly distributed random rotation numbers $\omega \in [0, \frac{1}{2}]$ computed by applying Hadamard's formula (5.12) to Φ_{500}. In figure 16 we exhibit the ratio $C_S(\omega) = e^{-2B(\omega)}/K(\omega)$ at the same values of ω: the ratio is clearly bounded away from 0 and $+\infty$ and has corners at the rationals, thus supporting the validity of our conjecture also for the SSM.

One might wonder now if one can improve lemma 3 so as to have $N_m(n) \leq [n/m]-1$, thus establishing that $1/n \log \delta_n \leq 2\sum_{k=0}^{+\infty}(\log q_{k+1})/q_k$ and the validity of (5.39) and (5.40). Yoccoz's proof of (2.8) does not make use of the majorant series method but rather analyses the behaviour of the Siegel radius under the action of $\omega \mapsto 1/\omega$ which is one of the two generators of the modular group and which also produces the continued fraction expansion. His proof is in some sense related to the renormalisation group approach (see, for instance, Mackay 1983) but exploits the analytic nature of the problem and makes heavy use of the uniformisation theorem for Riemann surfaces and distortion estimates for univalent functions. These powerful techniques are not available, however, for the study of real Hamiltonian systems and symplectic maps. Yoccoz himself therefore remarked that it would be interesting to know if the majorant

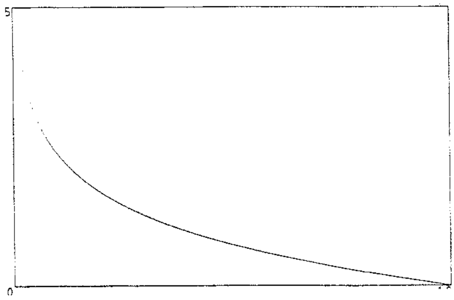

Figure 14. The width of the analyticity strip as a function of ε for the circle map (4.10) when $\omega = (\sqrt{5}-1)/2$.

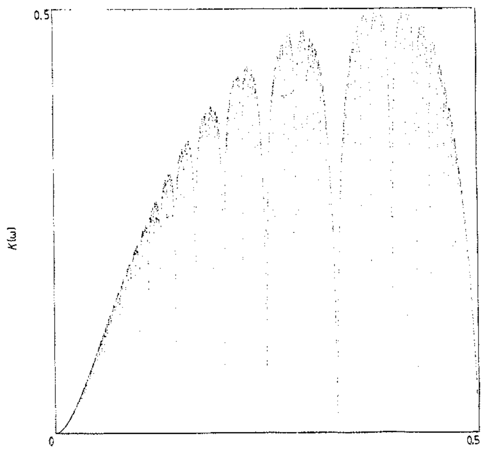

Figure 15. The critical function of the SSM at 5000 uniformly distributed random rotation numbers ω ; $[0, 1/2]$.

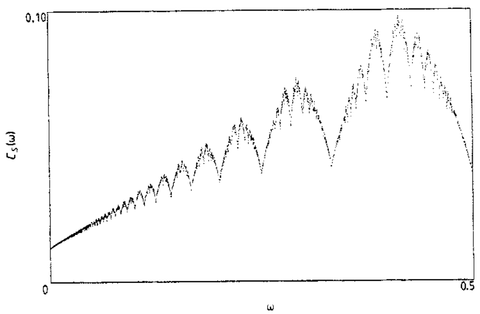

Figure 16. The ratio $C_S(\omega) := \exp(-2B(\omega)/K(\omega)$ for the SSM at the same ω of figure 15.

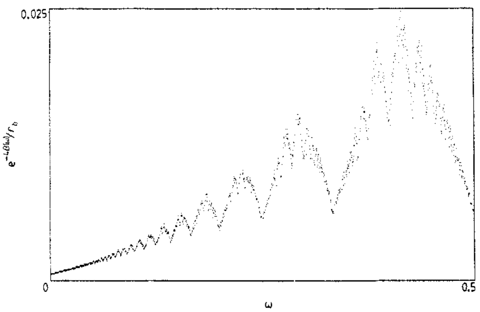

Figure 17. r_δ is the convergence radius of $\sum_{n=1}^{+\infty} \delta_n z^n$ computed by applying the Hadamard's formula to δ_{200}. The ratio $\exp(-4B(\omega))/r_\delta$ is computed at 2000 uniformly distributed random ω.

series method (in the version of Siegel and Brjuno) is, in fact, optimal with respect to the dependence of critical functions on the rotation number, i.e. if one can prove (2.6), (5.39) and (5.40) by that approach. This would also imply that the passage to absolute values does not affect the convergence radius of the series and would suggest the possibility of extending results of this kind to other, physically more interesting, problems where the majorant series can be used.

In order to attempt to answer this question we have computed numerically the first 200 terms of the sequence δ_n for the MSM at 2000 uniformly distributed random $\omega \in [0, 1/2]$. In figure 17 we exhibit the ratio between $\exp(-4B(\omega))$ and the radius of convergence of the series $\sum \delta_n z^n$ estimated by applying Hadamard's formula to δ_{200}. The result is clearly a function which is uniformly bounded away from 0 and $+\infty$, continuous with corners at rational ω, whereas the ratio with $\exp(-3B(\omega))$ or $\exp(-2B(\omega))$ diverges when ω is close to a rational.

Our numerical study suggests therefore that the majorant series method of Siegel and Brjuno cannot give the optimal dependence of critical functions on the rotation number, in the sense of allowing for a proof of our conjectures, even if the Brjuno condition is actually the optimal one for the problem of existence of analytic invariant curves , as it is proven in the appendix for the SSM.

On the other hand, KAM superconvergent methods do not use the continued fraction expansion of ω but only the coefficients appearing in Diophantine conditions, whereas the only other methods which exploit the algebraic nature and the scaling properties of the critical functions are the renormalisation group (Mackay 1983) and the modular smoothing technique (Buric *et al* 1989). We think that our study, even if this last negative result is discouraging as far as the goal of a complete rigorous characterisation of the critical functions is concerned, nonetheless shows that a rather complete analysis can be made, at least as far as one considers complex dynamical systems.

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Appendix. Divergence of the conjugation function for the SSM when the Brjuno condition is violated

Following section 5 the conjugation function Φ of the SSM verifies the recurrence

$$\Phi_1 = \frac{1}{D_1} \qquad \Phi_n = \frac{1}{D_n}\sum_{j=1}^{n-1}\frac{1}{j!}\sum_{k_1+\ldots+k_j=n-1}\Phi_{k_1}\ldots\Phi_{k_j} \qquad n \geq 2 \tag{A1}$$

where

$$D_n := |2\sin(\pi n\omega)|^2 \qquad n \geq 1 \tag{A2}$$

gives the small divisor sequence.

We remark that the following scaling relation holds:

$$\Phi_n = \left(\tfrac{1}{4}\right)^n \hat{\Phi}_n \qquad n \geq 1 \tag{A3}$$

where the new coefficients $\hat{\Phi}_n$ verify the recurrence

$$\hat{\Phi}_1 = \frac{1}{\varepsilon_1} \qquad \hat{\Phi}_n = \frac{1}{\varepsilon_n}\sum_{j=1}^{n-1}\frac{1}{j!}\sum_{k_1+\ldots+k_j=n-1}\hat{\Phi}_{k_1}\ldots\hat{\Phi}_{k_j} \qquad n \geq 2 \tag{A4}$$

and the small divisors are given by the same sequence ε_n used in the proof of the theorem of section 5:

$$\varepsilon_n := |\sin(\pi n\omega)|^2. \tag{A5}$$

Let ω be irrational but not verifying the Brjuno condition, i.e. $B(\omega) = +\infty$, or equivalently $\sum_{k=0}^{+\infty} (\log q_{k+1})/q_k = +\infty$. We denote by p_k/q_k its corresponding partial fractions, and $A := \{q_k \mid q_{k+1} \geq (q_k + 1)^2\}$. Since the Brjuno condition is violated by ω from (5.16) and (2.3) one clearly has that

$$\sum_{k=0}^{+\infty} \frac{1}{q_k} \log \frac{1}{\varepsilon_{q_k}} = +\infty. \tag{A6}$$

On the other hand, since $\varepsilon_k \geq q_{k+1}^2$,

$$\sum_{k\notin A} \frac{1}{q_k} \log \frac{1}{\varepsilon_{q_k}} < +\infty$$

and A must be infinite. If we denote the elements of A by $q_0' < q_1' < \ldots < q_k'$:

$$\sum_{k=0}^{+\infty} \frac{1}{q_k'} \log \frac{1}{\varepsilon_{q_k'}} = +\infty.$$

The divergence of the conjugation (A1) immediately follows from the following lemma.

Lemma 4. For every $k \geq 0$ one has the following estimate:

$$\frac{1}{q_{k+1}'} \log \hat{\Phi}_{q_{k+1}'} \geq a \left(\frac{1}{q_0'} \log \hat{\Phi}_{q_0'} + \sum_{j=0}^{k+1} \frac{1}{q_j'} \log \frac{1}{2\varepsilon_{q_j'}} \right) \tag{A7}$$

where a is a positive k-independent constant.

In fact from (A7) one has

$$\limsup_{n\to+\infty} \frac{1}{n} \log \Phi_n = \tfrac{1}{4} + \limsup_{n\to+\infty} \frac{1}{n} \log \hat{\Phi}_n = +\infty.$$

Proof of lemma 4. The proof is adapted from an argument given by Yoccoz (1988) for proving the divergence of the majorant series used for the quadratic map in the Siegel problem when the Brjuno condition is violated.

First of all we remark that the sequence $(\hat{\Phi}_n)_{n\in\mathbb{N}}$ is non-decreasing: $\hat{\Phi}_n \geq \hat{\Phi}_{n-1}/\varepsilon_n \geq \hat{\Phi}_{n-1}$. Moreover one can easily check by induction that for all i and s non-negative integers

$$\hat{\Phi}_{is-1} \geq \tfrac{1}{2} \hat{\Phi}_{s-1}^{i}. \tag{A8}$$

Let $k \geq 0$ and

$$n_k := \left(\frac{q_{k+1}'}{q_k' + 1} \right).$$

Applying (A8) to $\hat{\Phi}_{q'_{k+1}}$ one has

$$\hat{\Phi}_{q'_{k+1}} \geq \frac{1}{\varepsilon_{q'_{k+1}}} \hat{\Phi}_{q'_{k+1}-1} \geq \frac{1}{\varepsilon_{q'_{k+1}}} \hat{\Phi}_{n_k(q'_k+1)-1} \geq \frac{1}{2} \frac{1}{\varepsilon_{q'_{k+1}}} \hat{\Phi}^{n_k}_{q'_k}. \tag{A9}$$

Now, let $\alpha_k := n_k q'_k / q'_{k+1}$ and observe that the infinite product $\prod_{k=0}^{+\infty} \alpha_k < +\infty$ since $(q'_k + 1)^2 \leq q'_{k+1}$, which implies $\alpha_k \geq [1 - 1/(q'_k + 1)]^2$. Using this property of the sequence α_k one finds

$$\frac{1}{q'_{k+1}} \log \hat{\Phi}_{q'_{k+1}} \geq \frac{1}{q'_{k+1}} \log \frac{1}{2\varepsilon_{q'_{k+1}}} + \alpha_k \frac{1}{q'_k} \hat{\Phi}_{q'_k} \prod_{k=0}^{+\infty} \alpha_k \left(\frac{1}{q'_0} \log \hat{\Phi}_{q'_0} + \sum_{j=1}^{k+1} \frac{1}{q'_j} \log \frac{1}{2\varepsilon_{q'_j}} \right) .$$

The lemma is therefore proved, and $a := \prod_{k=0}^{+\infty} \alpha_k$. QED

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